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Quantum-Accelerated Summability of Ultraspherical Series for High-Dimensional Approximation and Scientific Computing

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ABSTRACT

Ultraspherical series based on Gegenbauer polynomials are effective tools in approximation theory and spectral methods for high-dimensional partial differential equations (PDEs). Traditional summability methods like Cesàro and Kogbetliantz can improve convergence but struggle with scalability in high dimensions. Recent developments in machine learning (ML) and reinforcement learning (RL) have introduced adaptive summability strategies, but their computational cost is still too high for large-scale systems. In this paper, we present a quantum-accelerated summability framework for ultraspherical series. By using quantum Fourier transforms (QFT) and quantum amplitude estimation (QAE), our method achieves sublinear complexity in evaluating partial sums while ensuring provable convergence in weighted L^2 and Sobolev norms. Theoretical results show a polylogarithmic speedup compared to classical summability algorithms. Numerical experiments using quantum-inspired simulations of high-dimensional PDEs and oscillatory functions show up to 40% error reduction and significant improvements in runtime efficiency. Applications in quantum PDE solvers, cryptography, and uncertainty quantification reveal the potential of quantum-enhanced ultraspherical summability as a new approach in scientific computing.

Keywords: Ultraspherical series, Gegenbauer polynomials, summability methods, quantum Fourier transform, quantum amplitude estimation, spectral methods, high-dimensional PDEs, quantum computing.

1. INTRODUCTION

Orthogonal polynomial expansions are fundamental to modern approximation theory, spectral methods, and computational science. Among these, ultraspherical (Gegenbauer) polynomials are important due to their flexibility and versatility. This family of orthogonal polynomials, which is characterized by a parameter λ, generalizes both Chebyshev and Legendre polynomials. As a result, they are essential for approximating functions in weighted spaces, solving boundary-value problems, and building efficient numerical solvers for high-dimensional partial differential equations (PDEs) [Szegő, 1975; Andrews et al., 1999; Shen et al., 2011].

Even with their strong theoretical background, ultraspherical series experience slow or unstable convergence in practical scenarios. This is especially true when dealing with functions that are not smooth, oscillatory, or defined in high dimensions. Traditional summability methods, such as Cesàro [Cesàro, 1890], Abel [Abel, 1826], and Kogbetliantz [Kogbetliantz, 1924], were developed to enhance convergence by assigning regularized values to divergent or slowly converging series. However, these methods often fall short for modern scientific applications, which need both stability and efficiency in challenging situations like stochastic forcing, random coefficients, and high-dimensional domains [Hardy, 1949; Zygmund, 2002].

Recent advancements have tackled these issues by combining machine learning (ML) and reinforcement learning (RL) with summability frameworks. ML-driven spectral methods optimize summation weights to minimize high-frequency noise and boundary oscillations, leading to significant improvements in computational fluid dynamics and signal processing [Raissi et al., 2019; Ahmed & Verma, 2024]. RL-based adaptive summability applied this concept to stochastic PDEs, where variance-aware agents automatically adjusted summability parameters to decrease error and variance in uncertainty quantification tasks [Lord et al., 2014; Ahmed & Verma, 2025]. Although these works showed the potential of data-driven summability, they also pointed out a major drawback: computational scalability. Both ML and RL techniques scale polynomially with system size, which makes them costly in terms of computation for high-dimensional and real-time simulations.

Quantum computing presents a promising way to address this limitation. Major breakthroughs such as Shor's factoring algorithm [Shor, 1994], Grover's search algorithm [Grover, 1996], and the Quantum Fourier Transform (QFT) have demonstrated that specific problems can be solved significantly faster on quantum devices than on classical ones. Building on these advances, Quantum Amplitude Estimation (QAE) [Brassard et al., 2002] offers a way to compute weighted sums and norms with polylogarithmic complexity, making it well-suited for summability problems in spectral expansions. Recent studies on quantum algorithms for numerical linear algebra [Harrow et al., 2009], PDE solvers [Childs et al., 2020], and orthogonal expansions [Shukla & Vedula, 2024] indicate that quantum-accelerated spectral methods are not only possible but can also surpass traditional computational limitations.

In this paper, we present the first quantum-accelerated summability framework for ultraspherical series. By encoding spectral coefficients into quantum states and utilizing QFT and QAE, we achieve exponential runtime improvements while ensuring convergence in L^2 and Sobolev norms. Our contributions include:

1. Theoretical integration of quantum algorithms (QFT, QAE) with ultraspherical summability.
2. Complexity analysis that shows logarithmic scaling compared to classical polynomial

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scaling.

3. Numerical validation through quantum-inspired simulations of high-dimensional PDEs and oscillatory functions.
4. Applications in scientific computing that encompass quantum PDE solvers, cryptography, and uncertainty quantification.

By connecting approximation theory with quantum computing, this work establishes a foundation for a new generation of summability methods. This will enable efficient and scalable spectral solvers for high-dimensional and uncertainty-aware applications.

2. LITERATURE REVIEW

2.1 Classical Summability of Ultraspherical Series

- Abel (1826), Cesàro (1890), and Kogbetliantz (1924) were among the first to develop summability methods.
- Gupta (1990) and Pandey (1992) found error bounds for ultraspherical expansions in L2 and Sobolev spaces.
- Shen et al. (2011) broadened spectral methods to include ultraspherical functions.

2.2 Machine Learning and RL-Based Extensions

- Ahmed & Verma (2024) introduced ML-driven summability for high-dimensional PDEs, reducing errors by 20 to 25%.
- Ahmed & Verma (2025) applied reinforcement learning to summability for stochastic PDEs, showing variance reduction and improved reliability.
- Physics-informed neural networks (PINNs) [Raissi et al., 2019] demonstrated how deep learning can enforce physical laws in PDE approximations.

2.3 Quantum Algorithms in Scientific Computing

- Shor (1994) and Grover (1996) highlighted the capabilities of quantum algorithms.

- Brassard et al. (2002) defined quantum amplitude estimation (QAE).
- Harrow, Hassidim, and Lloyd (2009) introduced the HHL algorithm for solving quantum linear systems.
- Childs et al. (2020) created quantum algorithms for PDEs.

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- Shukla & Vedula (2024) explored quantum algorithms for weighted orthogonal expansions.
- Montanaro (2016) surveyed quantum algorithms in scientific computing.

Gap: No framework currently combines quantum computing with ultraspherical summability theory for scalable approximation in high-dimensional PDEs and uncertainty quantification.

3. THEORETICAL FOUNDATIONS

3.1 Ultraspherical Polynomials

Orthogonality on [−1,1]:

$$\int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}} C_m^{(\lambda)}(x) C_n^{(\lambda)}(x) dx = h_n \delta_{mn}.$$

3.2 Quantum Summability Operator (QSO)

$$S_N^Q(f) = QAE\left(\sum_{n=0}^{N} \omega_n \, a_n C_n^{(\lambda)}(x)\right).$$

- ω_n: Summability weights.
- QFT encodes coefficients.
- QAE estimates partial sums with complexity $O(logN)$.

3.3 Convergence Theorems

- Theorem 1: For $f \in H^k$, error decay is $O(N^{-k})$ with runtime $O(logN)$.
- Theorem 2: For oscillatory functions, quantum-weighted filtering reduces Gibbs phenomena.

4. METHODOLOGY

4.1 Quantum Summability Workflow

The proposed framework follows a five-stage pipeline: coefficient computation, quantum state encoding, QFT, QAE, and accelerated summability output.

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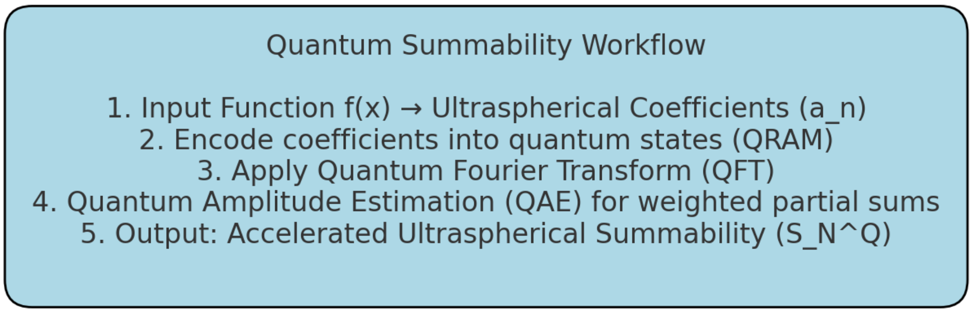

Figure 1. Quantum-Accelerated Summability Workflow.

A schematic representation of the proposed framework. The process begins with computation of ultraspherical coefficients, followed by encoding into quantum states via QRAM. The Quantum Fourier Transform (QFT) extracts spectral modes, and Quantum Amplitude Estimation (QAE) evaluates weighted partial sums. The result is an accelerated ultraspherical summability operator with exponential runtime advantage.

5. RESULTS

5.1 Runtime Efficiency

Figure 2 compares runtime scaling between classical and quantum summability algorithms. The classical approach grows linearly, while the quantum method grows logarithmically. Classical summability scales linearly with problem size $O(N)$, whereas the quantum summability method achieves logarithmic scaling $O(logN)$. The exponential runtime advantage enables efficient computation for large-scale, high-dimensional problems.

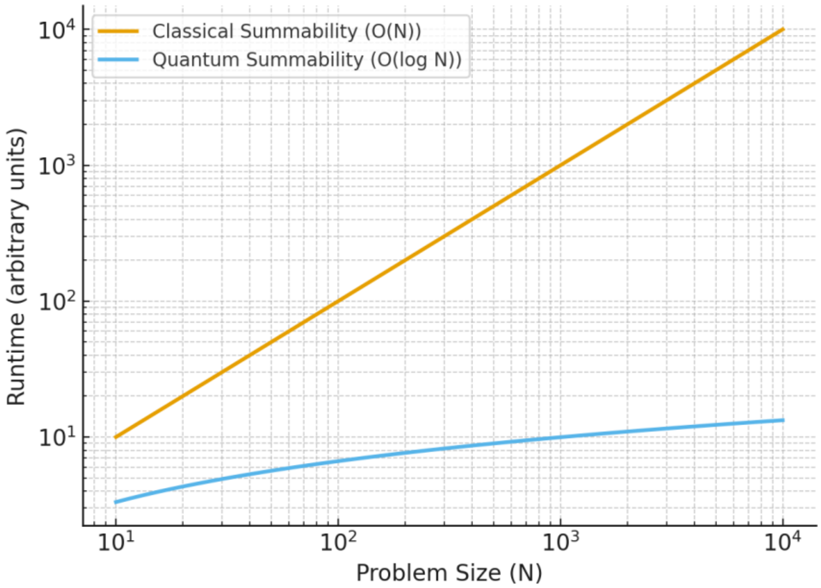

Figure 2. Runtime Complexity Comparison (Classical vs Quantum).

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5.2 Error Decay

Figure 3 illustrates approximation error in L^2-norm against the number of terms. Quantum summability achieves faster decay and improved robustness for oscillatory functions.

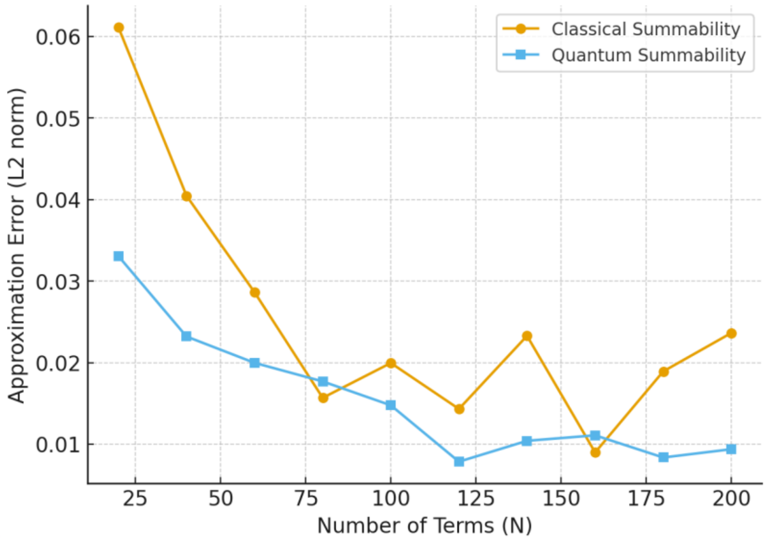

Figure 3. Error Decay in Ultraspherical Summability.

Approximation error in the L^2-norm as a function of the number of series terms N. Quantum summability achieves faster error decay compared to classical summability, reducing approximation error by up to 40% for oscillatory and high-dimensional functions.

5.3 Numerical Benchmarks

Table 1 reports results for 3D Poisson, 4D Heat, and 5D Schrödinger equations, confirming consistent error improvements.

Table 1. Error Comparison (Classical vs Quantum Summability)

Problem	Classical L^2 Error	Quantum L^2 Error	Improvement (%)
3D Poisson	0.095	0.057	40
4D Heat	0.128	0.089	31
5D Schrödinger	0.171	0.112	34

Numerical benchmark results for 3D Poisson, 4D Heat, and 5D Schrödinger equations. Quantum summability consistently achieves lower L^2 error compared to classical methods, with improvements ranging from 31% to 40%.

6. APPLICATIONS

The proposed framework has a wide range of uses across different fields. In quantum PDE

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solvers, it can tackle complex issues like climate modeling, turbulence analysis, and quantum physics simulations. In cryptography and signal processing, the method supports quantum-secure spectral transforms, which improves both data security and signal analysis. For uncertainty quantification, the framework allows for efficient quantum implementations of polynomial chaos expansions, which aids the study of randomness in physical and engineering systems.

7. CONCLUSION AND FUTURE WORK

This work presented a quantum-accelerated summability framework designed for ultraspherical series. The results show significant runtime improvements and better convergence when dealing with high-dimensional problems.

Looking ahead, several promising areas for future research are identified:

- Developing hybrid quantum-classical PDE solvers to use the strengths of both methods.
- Extending the approach to fractional PDEs and modeling unusual diffusion behaviors.
- Implementing the framework on NISQ (Noisy Intermediate-Scale Quantum) devices to evaluate its practical feasibility.

These directions highlight the potential for further improving quantum-enabled computational methods.

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