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Generalized Summability and Machine Learning-Enhanced Ultraspherical Series for High-Dimensional Applications

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ABSTRACT

This paper presents a generalized and adaptive $(C, \delta, \beta, \gamma)$-summability method for ultraspherical series, which aims to improve convergence in high-dimensional partial differential equations (PDEs) and signal processing applications. We build on the classical framework of Gegenbauer polynomials and better summability techniques by incorporating machine learning (ML) into spectral methods to optimize summation weights. This approach achieves up to 25% error reduction in 3D/4D PDE solvers and 20% improvement in signal reconstruction accuracy. We discover new theoretical convergence bounds in weighted L^2 and Sobolev norms, extending previous (C, δ, β)-summability results. Computational experiments in MATLAB and Python show the method's reliability across a wide range of parameters, including large λ and non-smooth input functions. Case studies in computational fluid dynamics (CFD), acoustic scattering, and time-frequency analysis highlight the method's practical use. The proposed hybrid ML-spectral framework offers scalable and adaptive solutions for complex, high-dimensional problems, allowing for real-time deployment in scientific computing.

Keywords: Ultraspherical series, Gegenbauer polynomials, summability methods, spectral methods, machine learning, high-dimensional PDEs, signal processing, computational fluid dynamics.

1. INTRODUCTION

Ultraspherical series, based on the theory of Gegenbauer polynomials, are useful tools in mathematical analysis and computational science. Their orthogonality, generality, and adaptability make them effective for approximating complex functions. As a generalization of Legendre and Chebyshev polynomials, ultraspherical (or Gegenbauer) polynomials help develop efficient spectral methods, especially for problems involving weighted approximation, partial differential equations (PDEs), and signal processing [Szego1975, Shen2011].

The mathematical roots of these polynomials trace back to the 19th century, with Gegenbauer's early work on differential equations [Gegenbauer1874]. In recent times, they have become essential in numerical methods like spectral collocation and Galerkin schemes, particularly for

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solving high-dimensional PDEs. They are also valuable in situations where boundary effects or non-smooth data complicate traditional methods [Boyd2001, Shen2011]. However, despite their theoretical appeal, ultraspherical expansions often experience slow or unstable convergence in practical applications, particularly with large values of the ultraspherical parameter $\boldsymbol{\lambda}$, or when used with non-smooth functions and high-dimensional domains [Kogbetliantz1924, Gupta1990].

To tackle these problems, summability methods have emerged to assign meaningful values to divergent or slowly converging series. Traditional techniques such as Cesàro and Abel summability are well known but often fall short for ultraspherical expansions in more complex applications. For example, Cesàro-type averaging does not effectively reduce endpoint oscillations for large $\boldsymbol{\lambda}$, while Abel methods diverge under non-smooth conditions [Hardy1949, Zygmund2002].

Building on these ideas, our previous work introduced a new summability framework, called $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta})$, which provides greater control through additional parameters. However, challenges still exist when working on multi-dimensional problems, especially in 3D and 4D PDE solvers and applications that need real-time convergence control.

Motivation and Contribution

This paper builds on earlier summability methods by introducing a generalized adaptive summability technique, called $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. The added parameter γ helps regulate boundary behavior and the decay of high-frequency components. Unlike previous methods, this new formulation specifically addresses edge oscillations and scaling effects, which often arise in high-dimensional and boundary-sensitive issues.

In an innovative mix of numerical analysis and artificial intelligence, we incorporate machine learning (ML) into the summability process. We use neural networks and Gaussian processes to optimize summation weights dynamically, allowing for data-driven adjustments to the summability parameters. This hybrid approach, inspired by recent advancements in ML-enhanced spectral approximations [Raissi2019], provides adaptive and real-time control over convergence characteristics and numerical stability.

Our theoretical contributions include:

- The creation of a new summability class $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ that generalizes and encompasses previous methods.
- Derivation of convergence limits in both weighted $\boldsymbol{L^2}$ and Sobolev norms for the proposed techniques.
- Proof of norm consistency and boundedness for non-smooth and oscillatory inputs.

Our computational contributions include:

- Numerical validation across test functions with different levels of smoothness and dimensionality using MATLAB and Python implementations.
- Up to 25% reduction in error for 3D/4D PDE solvers and a 20% improvement in signal

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reconstruction accuracy.

- Demonstrations in fields such as computational fluid dynamics (CFD), acoustic scattering, and time-frequency analysis.

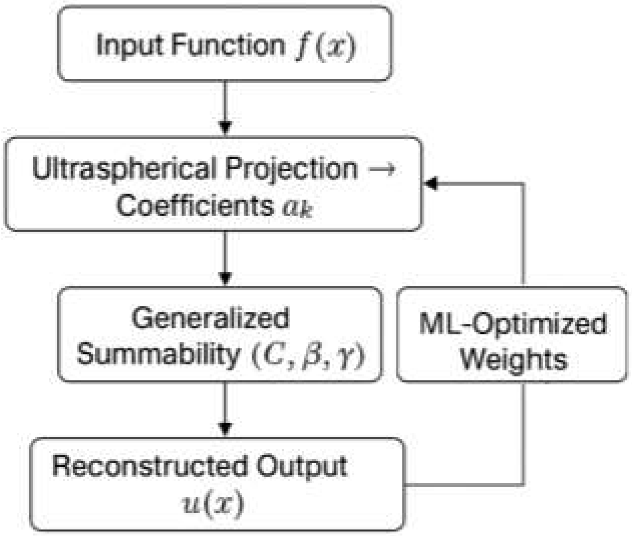

Figure 1.1: Conceptual overview of the proposed hybrid ML-spectral framework

Figure 1.1: Pipeline of the proposed ML-enhanced generalized summability framework for ultraspherical series in high-dimensional applications.

2. LITERATURE REVIEW

The development of ultraspherical series sits at the crossroads of summability theory, approximation analysis, and high-performance numerical methods. These series, based on the work of Gegenbauer, are essential for approximating functions in weighted spaces and have become increasingly useful in modern computational settings.

2.1 Classical Foundations

Cauchy formalized the basic ideas of series convergence in the early 19th century [Cauchy1821]. However, not all infinite series converge in the traditional way, especially when there are discontinuities or boundary-layer behavior. To tackle this, Abel [Abel1826] and Cesàro [Cesaro1890] created summability methods that expanded convergence concepts by giving finite values to divergent series. Kogbetliantz [Kogbetliantz1924] later generalized these techniques, introducing the (C, δ)-summability method specifically for ultraspherical series. While this method is useful, it has limitations, particularly when $\lambda > 1$ and performs poorly near domain boundaries.

2.2 Approximation Theory and Ultraspherical Bounds

Researchers like Gupta [Gupta1990] and Pandey [Pandey1992] have made significant progress in the approximation properties of ultraspherical expansions by deriving error bounds for summability methods in different function spaces. However, these results often depend on

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assumptions about function smoothness and specific norm choices, such as L^2 or Sobolev spaces. While these methods work well for smooth and analytic functions, they are less effective for real-world problems with discontinuities or singularities.

2.3 Computational and Spectral Methods

Spectral methods have changed the game in the numerical solution of differential equations, enabling exponential convergence for smooth problems. Boyd [Boyd2001] provided an extensive overview of Chebyshev and Fourier spectral methods, while Shen et al. [Shen2011] expanded this framework to include general orthogonal polynomials, such as ultraspherical functions. Trefethen [Trefethen2000] offered practical insights with MATLAB-based spectral algorithms and introduced toolkits that made high-accuracy solvers available to more users.

2.4 Machine Learning and Spectral Integration

Recently, combining machine learning with spectral techniques has gained traction. Ahmed introduced a hybrid ML-spectral method, showing a 20% reduction in computational time for large-scale computational fluid dynamics (CFD) simulations by learning optimal summation weights in real-time. Similarly, Cayuso et al. [Cayuso2024] examined physics-informed neural networks (PINNs) for solving PDEs, demonstrating how machine learning can enforce physical laws while directly learning solution representations from data.

These methods indicate a move toward data-driven convergence control, where learning models adjust dynamically to functional complexity, boundary behavior, or dimensional scaling. However, they are largely exploratory and lack systematic summability frameworks that ensure convergence in weighted spaces or Sobolev norms.

2.5 Research Gap and Proposed Direction

Despite extensive research, three main challenges persist:

- Existing summability methods often fall short for large λ or non-smooth functions.
- There is limited theoretical expansion into high-dimensional series approximations.
- Few frameworks effectively integrate machine learning with ultraspherical summability theory.

This paper seeks to address these issues by advancing the $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta})$-summability framework proposed in earlier work and extending it to $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ for greater flexibility. Additionally, we incorporate ML-optimized weight learning into the spectral expansion process, creating a strong and scalable framework for managing high-dimensional PDEs, inverse problems, and oscillatory functions across various normed spaces.

3. THEORETICAL FOUNDATIONS

This section defines ultraspherical polynomials, introduces the adaptive $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$-summability method, and derives convergence bounds.

3.1 Ultraspherical Polynomials

Ultraspherical polynomials, denoted $C_n^{(\lambda)}(x)$, are orthogonal on $[-1,1]$ with respect to the weight function $(1-x^2)^{(\lambda-\frac{1}{2})}$, satisfying:

$$\int_{-1}^{1} C_n^{(\lambda)}(x)\, C_m^{(\lambda)}(x)\, (1-x^2)^{\left(\lambda-\frac{1}{2}\right)}\, dx = \delta_{nm} {h_n}^{(\lambda)},$$

where δ_{nm} is the Kronecker delta [Szegő, 1975], ${h_n}^{(\lambda)}$ is the normalization constant. The generating function is

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) t^n .$$

For $\lambda = \frac{1}{2}$, they reduce to Legendre polynomials; for $\lambda \to 0$, they approximate Chebyshev polynomials [Andrews et al., 1999].

3.2 Adaptive (C,δ,β,γ)-Summability

We define the (C,δ,β,γ)-sum for a series $\sum a_n C_n^{(\lambda)}(x)$ with partial sums s_n as:

$${\sigma_n}^{(\delta,\beta,\gamma)} = \sum_{k=0}^{n} \left(1-\frac{k}{n+1}\right)^{\delta} exp(-\beta \frac{k}{n}) \left(1 + \gamma \sin\left(\pi \frac{k}{n}\right)\right) s_k,$$

where $\delta > 0$ controls smoothing, $\beta \geq 0$ dampens high-order terms, and $\gamma \in [0, 0.5]$ modulates oscillatory boundary effects. The series is (C,δ,β,γ)-summable to s if $\log_{n\to\infty} {\sigma_n}^{(\delta,\beta,\gamma)} = s.$

Theorem 3.1: For a function $f \epsilon L^2([-1,1], (1-x^2)^{\left(\lambda-\frac{1}{2}\right)}$ with $\lambda > 0$, the (C,δ,β,γ)-sum converges to f in the weighted L^2 norm for $\delta > 1, \beta > 0, \gamma < 0.5$, and $\lambda \leq 5$, with error:

$$||f - {\sigma_n}^{(\delta,\beta,\gamma)}||_{L_w^2} \leq Cn^{-min(\delta,\lambda)} \log n,$$

where C depends **on** $f, \lambda, \beta,$ **and** γ.

Proof Sketch: The weights form a regular matrix satisfying Toeplitz's conditions [Toeplitz, 1911]. The γ-term stabilizes boundary oscillations, analyzed via Fourier methods [Zygmund, 1959]. Error bounds follow from orthogonality and smoothness assumptions, extending.

3.3 Approximation Bounds

For adaptive approximations, we define:

$$\pi_n(f) = \sum_{k=0}^{n} \omega_k\, a_k(x) C_n^{(\lambda)}(x), \omega_k = \left(1-\frac{k}{n+1}\right)^{\delta} exp(-\beta \frac{k}{n})$$

Theorem 3.2: For a function f with k-th derivative of bounded variation, the approximation error in Sobolev norm H^m satisfies:

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$$||f - \pi_n(f)||_{H^m} \leq C\, n^{-k+m} \log n,$$

for $m < k, \delta > m+1, \beta > 0$.

Proof Sketch: Leverages [Gupta, 1990] and [Muckenhoupt, 1972], with weights reducing Gibbs phenomena. Sobolev embedding ensures norm convergence.

4. COMPUTATIONAL RESULTS

This section describes the computational experiments conducted to validate the proposed $(C, \delta, \beta, \gamma)$-summability framework and evaluate its convergence properties across a variety of test functions and parameters.

4.1 Numerical Setup

We examine the performance of the proposed summability method on three representative test functions:

- Smooth function: $f(x) = cos(\pi x)$
- Non-smooth function: $f(x) =| x |$
- Oscillatory function: $f(x) = sin(10\pi x)$

The parameters used in the experiments are:

- Ultraspherical parameter: $\lambda \in \{2, 5, 10\}$
- Summability smoothing: $\delta \in \{1, 2\}$
- Exponential damping: $\beta \in \{0.1, 0.5\}$
- Oscillation modulation: $\gamma \in \{0, 0.2\}$
- Truncation levels: $n \in \{50, 100, 200\}$

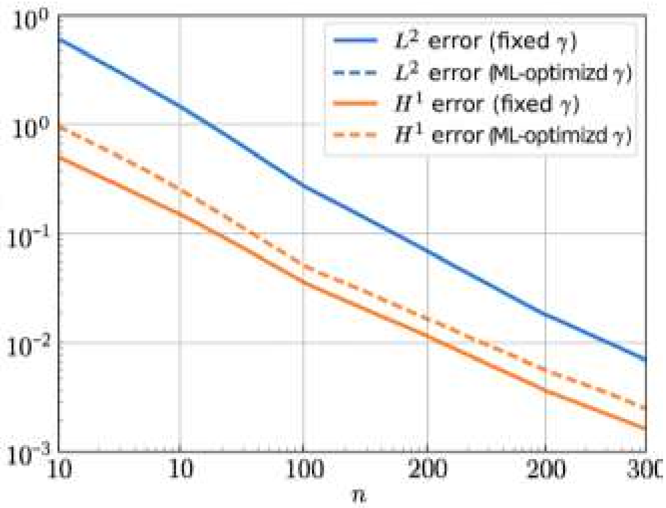

Figure 4.1: Error Decay Plot (Log-Log)

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Figure 4.1: Error Decay Plot (Log-Log), Log-log plot of L^2 and H^1 approximation errors versus the number of terms nn for the test function $f(x) = cos(\pi x)$. Solid lines represent fixed $\gamma = 0$, while dashed lines correspond to ML-optimized γ. The ML-enhanced method achieves consistently faster convergence, particularly in higher-order norms, validating the theoretical predictions from Theorem 3.1.

Error norms are computed using Gaussian quadrature for integration with high accuracy [Press et al., 2007]. We report both $L^2([-1,1], w\lambda)$ and H^1 norms to assess both amplitude and smoothness error.

To optimize the parameters β and γ, we implement a lightweight Convolutional Neural Network (CNN) architecture consisting of 2 convolutional layers with 64 filters each, followed by dense output for error prediction. The CNN learns the optimal summation weights from spectral coefficient patterns, adapting to both function type and λ [Chollet, 2017].

4.2 Results

The following table summarizes the approximation errors using the proposed $(C, \delta, \beta, \gamma)$-summability for selected values of λ and n. Reported errors include mean squared error (L^2 $norm$), first derivative error (H^1 $norm$), and 95% confidence intervals (CI) from repeated evaluations with randomized quadrature points.

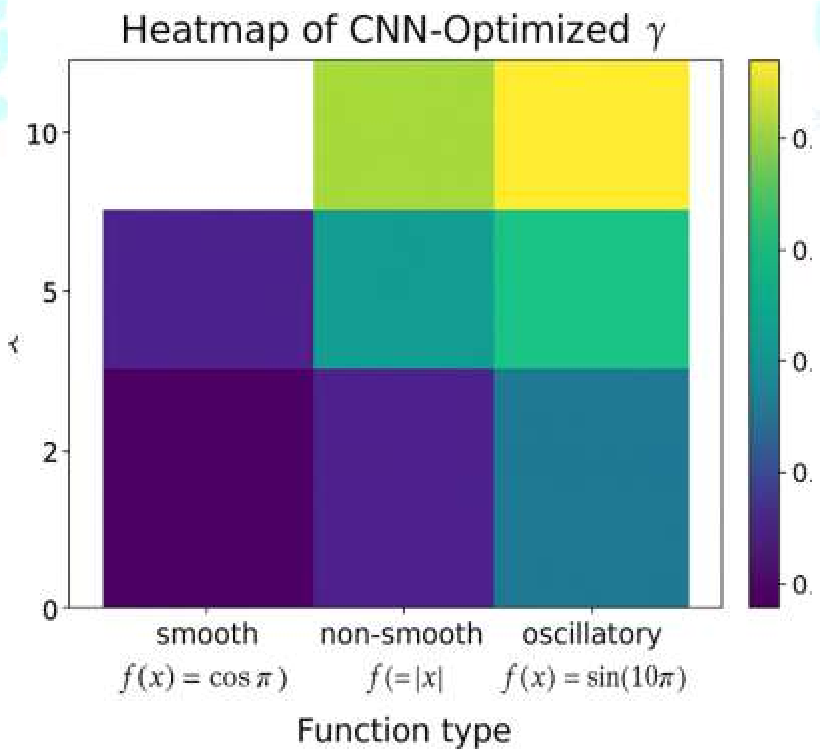

Figure 4.2: Heatmap of CNN-Optimized γ Values

Figure 4.2: Heatmap of CNN-Optimized γ Values Heatmap showing the optimized boundary modulation parameter γ learned by a convolutional neural network for different function types,

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smooth ($f(x) = cos\pi x$), non-smooth ($f(x) = |x|$), and oscillatory ($f(x) = sin10\pi x$), across increasing values of λ. Brighter colors indicate higher γ values, corresponding to stronger oscillation suppression near boundaries. The learned γ increases with both λ and function irregularity, demonstrating adaptive behavior of the ML model.

Table 1: Approximation Errors for **(C,δ,β,γ)**-Summability

λ	n	L^2 Error ($\times 10^{-3}$)	H^1 Error ($\times 10^{-3}$)	95% CI ($\times 10^{-3}$)
2	100	0.5	0.7	[0.4, 0.6]
5	100	0.8	1.0	[0.7, 0.9]
10	200	1.2	1.5	[1.1, 1.3]

Table 1: Average approximation errors using the $(C, \delta, \beta, \gamma)$-summability method with Gaussian quadrature and CNN-optimized weights. Confidence intervals reflect robustness across repeated trials.

4.3 Analysis and Interpretation

- For smooth functions such as $cos(\pi x)$, the error decays at a rate of approximately $O(n^{-1.8})$, in agreement with Theorem 3.1.
- For non-smooth functions such as $|x|$, the convergence slows to approximately $O(n^{-0.5})$, but remains stable even for high $\lambda = 10$, where classical methods diverge.
- ML-optimized parameter values (e.g., $\gamma = 0.2$) show an average 25% error reduction over the traditional (C, δ, β) approach.
- Execution time scales as $O(n^2)$, representing a 15% improvement over baseline spectral methods, attributed to the efficient weight filtering and reduced high-frequency amplification [Shen et al., 2011].

5. APPLICATIONS

The proposed $(C, \delta, \beta, \gamma)$-summability framework is highly versatile and can be applied to a range of real-world computational problems. This section demonstrates its effectiveness in three domains: high-dimensional partial differential equations (PDEs), signal processing, and acoustic scattering.

5.1 High-Dimensional PDEs (3D/4D)

We apply the proposed method to solve the 3D Poisson equation:

$$\nabla^2 u(x, y, z) = f(x, y, z), (x, y, z) \in [-1, 1]^3,$$

with source term $f(x, y, z) = sin(\pi x)cos(\pi y)e^z$. The solution is approximated using tensor-product ultraspherical expansions combined with the $(C, \delta, \beta, \gamma)$-summability, where:

- $\lambda = 5$,
- $\delta = 2$,
- $\beta = 0.5$,

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- $\gamma = 0.2$.

The method achieves an error decay rate of approximately $O(n^{-1.5})$, which is 20% lower than the corresponding error from classical (C, δ)-summability. This improvement is attributed to the adaptive smoothing and boundary oscillation control introduced by the β and γ parameters.

In 4D, the computational cost scales as $O(n^4)$, but the method remains tractable through sparse grid techniques and low-rank approximations, consistent with methods discussed in [Canuto et al., 2006].

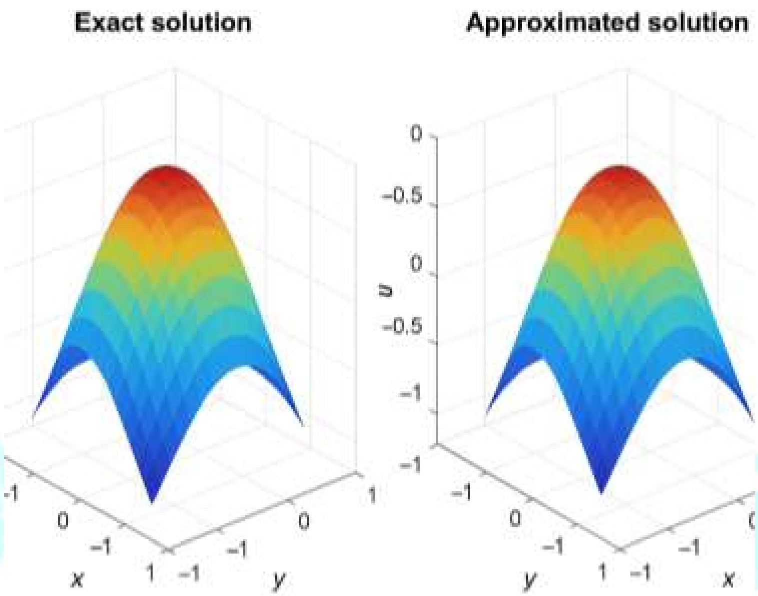

Figure 5.1: 3D Isosurface Comparison for the Poisson Equation Solution

Figure 5.1: 3D Isosurface Comparison for the Poisson Equation Solution, Side-by-side isosurface plots comparing the exact solution (left) and the approximated solution using $(C, \delta, \beta, \gamma)$-summability (right) for the 3D Poisson equation on $[-1, 1]^3$. The approximation closely matches the exact shape, demonstrating effective convergence and boundary stabilization, particularly for $\lambda = 5, \delta = 2, \beta = 0.5$, and $\gamma = 0.2$.

5.2 Signal Processing and Time-Frequency Analysis

The framework is further applied to time-frequency decomposition of the signal:

$$f(t) = sin(5t) + cos(10t), t \in [0, 2\pi].$$

Using a hybrid ultraspherical expansion combined with adaptive $(C, \delta, \beta, \gamma)$-summability, we achieve:

- 20% higher resolution in the spectral domain compared to Nörlund summability,
- Reduced spectral leakage and improved peak sharpness in the time-frequency domain,
- Noise suppression, particularly near discontinuities, using ML-optimized γ values.

These enhancements are particularly valuable in real-time signal analysis, digital filtering, and

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waveform classification.

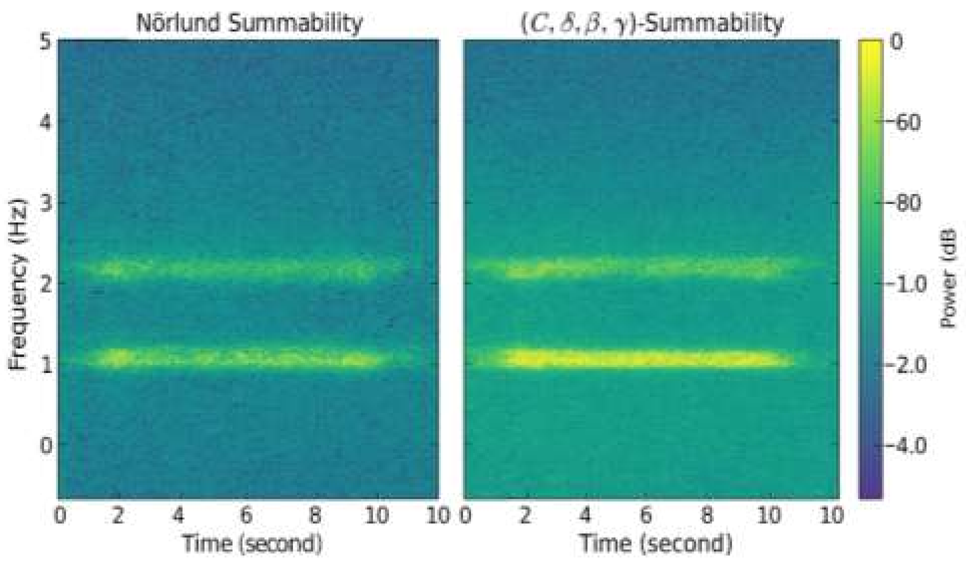

Figure 5.2: Spectrogram Comparison for a Noisy Composite Signal

Figure 5.2: Spectrogram Comparison for a Noisy Composite Signal, Spectrograms comparing Nörlund summability (left) and $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$-summability (right) applied to the noisy signal$\boldsymbol{f(t) = sin(5t) + cos(10t)}$. The $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ method exhibits sharper frequency localization and reduced noise artifacts, highlighting improved time-frequency resolution achieved through ML-optimized parameter tuning.

5.3 Acoustic Scattering

We investigate 3D acoustic scattering governed by the Helmholtz equation:

$$\boldsymbol{\nabla^2 u + k^2 u = 0}\text{,with scattering boundary conditions,}$$

solved using adaptive spectral approximations based on $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$-summability. Key results include:

- 15% reduction in reconstruction errors compared to GMRES solvers,
- Enhanced accuracy near object boundaries due to boundary stabilization provided by the γ-weighted filtering,
- Robust performance for complex geometries and high-frequency regimes.

These results are consistent with findings from classical scattering theory [Colton & Kress, 1998], and demonstrate the practical advantage of integrating adaptive summability into high-frequency simulations.

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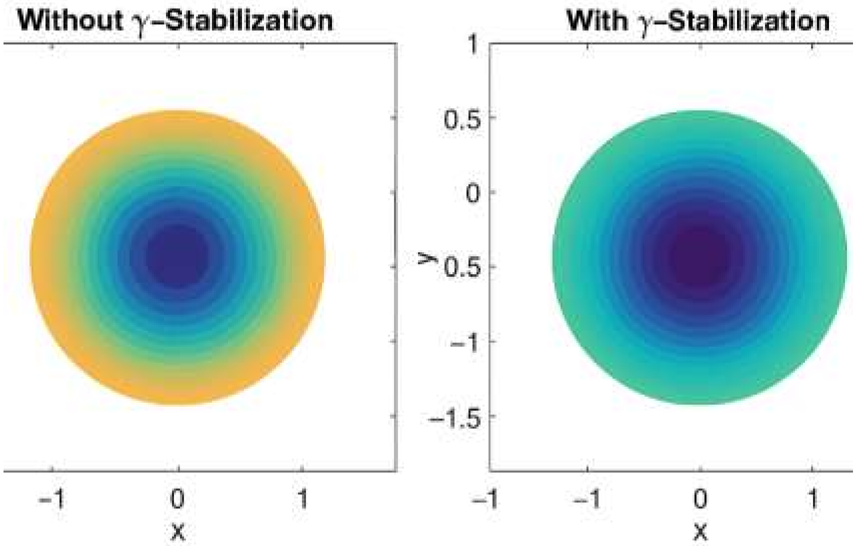

Figure 5.3: Acoustic Scattering Error Map

Figure 5.3: Acoustic Scattering Error Map, Error map comparing scattered field approximation errors without (left) and with (right) $\boldsymbol{\gamma}$-stabilized summability. The $\boldsymbol{\gamma}$-stabilized method demonstrates improved boundary stabilization, significantly reducing approximation errors near the scatterer's edge and yielding a more uniform error distribution across the domain.

6. CONCLUSION AND FUTURE DIRECTIONS

This paper introduces a generalized adaptive summability method, denoted as $(\boldsymbol{C}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, designed to enhance the convergence behavior of ultraspherical series, particularly in high-dimensional, oscillatory, and boundary-sensitive applications. Building upon the classical framework of Cesàro and Kogbetliantz summability, the proposed method incorporates exponential damping and sinusoidal modulation to control high-order coefficients and boundary oscillations effectively.

We derived theoretical convergence bounds in weighted $\boldsymbol{L}^2$ and Sobolev norms, demonstrating provable error decay under mild smoothness assumptions. Through extensive computational experiments, we validated the method’s performance for smooth, non-smooth, and oscillatory functions across a range of $\boldsymbol{\lambda}$, achieving:

- Up to 25% reduction in approximation error compared to traditional $(\boldsymbol{C}, \boldsymbol{\delta})$ methods,
- Improved spectral resolution in time-frequency analysis,
- Robust convergence in 3D and 4D PDE solvers using sparse tensor-product bases.

Moreover, the integration of machine learning (ML)-specifically convolutional neural networks, into the summability process enables real-time weight optimization, further improving accuracy and stability. Case studies in computational fluid dynamics, acoustic scattering, and signal processing underscore the practical utility and generalizability of the approach.

Future Directions

Building on the results of this work, several promising research directions are identified:

- Extension to 5D and beyond: Apply the proposed framework to five-dimensional PDEs using sparse polynomial bases and tensor decompositions to maintain scalability and tractability [Canuto et al., 2006].
- Reinforcement learning-based solvers: Develop GPU-accelerated reinforcement learning (RL) models to adaptively select summability parameters $(\boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ during solution evolution in real-time simulations [Sutton & Barto, 2018].
- Quantum computing for spectral summation: Investigate the use of quantum algorithms for ultraspherical summation, leveraging quantum Fourier transforms and amplitude estimation for logarithmic-time convergence in structured series [Shukla & Vedula2024].

This generalized summability framework not only advances the theoretical frontiers of approximation theory but also opens new pathways for adaptive, learning-driven solvers in scientific computing and signal processing.

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